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BUCKLING OF STRUCTURES WITH FINITE PREBUCKLING DEFORMATIONS—A PERTURBATION, FINITE ELEMENT ANALYSIS

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Abstract—In some technically important structures, finite prebuckling displacements have a profound effect on the bifurcation load. To ignore these displacements, as is done in most instability analyses, is to invite major errors, usually on the unsafe side. A method is presented which approximates this effect without the necessity of solving nonlinear equations. The general theory is developed for any elastic body under conservative loads. The governing equations are subsequently discretized by a finite element approach and it is shown that for planar framed structures, the second order approximation to the buckling load can be found in terms of the standard linear and geometric stiffness matrices of structural analysis; the solution procedure does not require iterations. For illustrative purposes, a computer program was developed for planar structures and the results are compared to the exact solution for the buckling of shallow circular arches.

1. INTRODUCTION

In the static approach to the solution of bifurcation problems in stability, one determines whether or not more than one configuration of the structure exists under a given load. For simplicity, let this set of loads be characterized by a single load parameter λ, so that a path can be drawn in load-deflection space. As the load is increased, the relationship between this single load parameter and the corresponding displacements define a fundamental path. The bifurcation point is then identified by non-uniqueness of the fundamental path in the neighborhood of known equilibrium solutions. Mathematically, bifurcation is associated with a loss of the positive definiteness of the total potential energy in an infinitesimal excursion from equilibrium, provided that the external loads are conservative.

The equations governing the stability problem are obtained by examining the terms of the total potential energy which are quadratic in these "additional" displacements or their derivatives. The general theory set forth by Koiter[1] and later reworked by Budiansky[2] is a perturbation technique for finding other solutions to the equilibrium equations in the neighborhood of the fundamental path.

Typically, the prebuckling displacements are neglected in this procedure. This works well for many engineering applications. However, there are technically important examples where this is not the case. Neglecting the prebuckling displacements often leads to errors which are not on the conservative side.

The displacements prior to and at buckling may be expanded in a power series in the loading parameter λ. Thompson[3] and Kerr and Soifer[4] have demonstrated that serious errors can result when the fundamental path is linearized. The most desirable approach is to superpose the additional, infinitesimal displacements of the buckling mode onto the possibly large displacements prior to buckling. However, this, as expected, leads to computational complexities. Fitch[5], and independently Cohen[6], have further generalized the work of Budiansky[2] so that a linearized prebuckling state and negligible prebuckling displacements need not be assumed. Cohen and Haftka[7] attempted to retain these features and yet keep the efficiency of a linear analysis with their "modified structure" approach. The method, based on Koiter's imperfection theory, was successful in simple nonlinear cases. Gallagher[8] has employed a discrete analysis using a piecewise-linear fundamental path.

Masur and Schreyer[9], on the other hand, have developed a method for incorporating the effects of prebuckling displacement directly, without a progressive iterative solution. The computational effort in this procedure is reduced considerably. The method utilizes a power series expansion of the prebuckling state as well as of the buckling parameter and mode; hence the degree of accuracy is dependent on the number of terms retained. The inclusion of prebuckling displacements along a general fundamental path to the point of bifurcation without mathematical complexities results in improved solutions to problems where the prebuckling state is not trivial. In the present investigation, the method of Masur and Schreyer is applied to a discrete model analysis.

2. DEVELOPMENT OF PRE-BUCKLING EQUATIONS

An elastic body occupying a volume τ bounded by S is deformed so that a generic point P in the reference state displaces to P^*. The stability of the deformed configuration is to be investigated. Let a_i be the Cartesian coordinates of the reference state and let X_i be the Cartesian coordinates of the deformed state. The displacement field is given by

$$u_i = X_i - a_i. \tag{1}$$

In what follows, derivatives are with respect to the original state, and are indicated by $(\ )_{,i} = \partial(\ )/\partial a_i$, while a repeated index will imply a summation.

The Green strain tensor is given by

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \tag{2}$$

while the corresponding Kirchhoff–Piola stress tensor of the second kind is

$$S_{ik} = J\left(\frac{\partial a_i}{\partial X_j}\right)\left(\frac{\partial a_k}{\partial X_l}\right)\sigma_{jl} \tag{3}$$

where J is the Jacobian for the transformation between the current and original coordinates, and σ_{ij} is the physical (Cauchy) stress tensor. The equations of equilibrium are

$$S_{ij,j} + (S_{kj}u_{i,j})_{,k} = 0 \quad \text{in } \tau, \tag{4}$$

and the associated boundary conditions for the problems considered here are

$$S_{ij}n_j + S_{kj}u_{i,j}n_k - \lambda T_i = 0 \quad \text{on } S_T \tag{5}$$

and

$$u_i = 0 \quad \text{on } S_u. \tag{6}$$

Here n_i is the unit normal in the initial state, T_i are applied surface tractions, S_T and S_u are the traction-specified surface and displacement-specified surface, respectively, and λ is a multiplicative load parameter. Following Masur and Schreyer[9] we assume that the equilibrium stress S_{ij} and the corresponding displacement field u_i can be expanded near the unstressed state in terms of the load parameter λ, in the form

$$\begin{aligned} S_{ij} &= \lambda S_{ij}^{(1)} + \lambda^2 S_{ij}^{(2)} + \cdots \\ u_i &= \lambda u_i^{(1)} + \lambda^2 u_i^{(2)} + \cdots. \end{aligned} \tag{7}$$

If we substitute eqn (7) into eqns (4)–(6), respectively, we obtain

$$\lambda\{S_{ij,j}^{(1)}\} + \lambda^2\{S_{ij,j}^{(2)} + (S_{kj}^{(1)}u_{i,j}^{(1)})_{,k}\} + \cdots = 0 \quad \text{in } \tau \tag{8}$$

$$\lambda\{S_{ij}^{(1)}n_j - T_i\} + \lambda^2\{S_{ij}^{(2)}n_j + S_{kj}^{(1)}u_{i,j}^{(1)}n_k\} + \cdots = 0 \quad \text{on } S_T \tag{9}$$

and

$$\lambda u_i^{(1)} + \lambda^2 u_i^{(2)} + \cdots = 0 \quad \text{on } S_u. \tag{10}$$

Since eqns (8)–(10) are valid for all values of λ, the coefficient of each power of λ must vanish individually. The following stress equations then are obtained

$$\begin{aligned} &S_{ij,j}^{(1)} = 0 \\ &S_{ij,j}^{(2)} = -(S_{kj}^{(1)} u_{i,j}^{(1)})_{,k} \quad \text{etc.,} \quad \text{in } \tau \end{aligned} \tag{11}$$

which are associated with the boundary conditions

$$\begin{aligned} &S_{ij}^{(1)} n_j - T_i = 0 \\ &S_{ij}^{(2)} n_j + S_{kj}^{(1)} u_{i,j}^{(1)} n_k = 0 \quad \text{etc.,} \quad \text{on } S_T \end{aligned} \tag{12}$$

and

$$\begin{aligned} &u_i^{(1)} = 0 \\ &u_i^{(2)} = 0 \quad \text{etc.,} \quad \text{on } S_u. \end{aligned} \tag{13}$$

We will use the elastic constitutive relation

$$S_{ij} = C_{ijkl}\epsilon_{kl}. \tag{14}$$

By eqn (2) and the symmetry of the stress tensor, eqn (14) becomes

$$S_{ij} = C_{ijkl}(u_{k,l} + 1/2 u_{m,k} u_{m,l}). \tag{15}$$

If the elastic stress–strain law is written in terms of the stresses and displacements expanded in λ, as given in eqn (7), it follows that

$$\begin{aligned} &S_{ij}^{(1)} = C_{ijkl} u_{k,l}^{(1)} \\ &S_{ij}^{(2)} = C_{ijkl}(u_{k,l}^{(2)} + 1/2 u_{m,k}^{(1)} u_{m,l}^{(1)}), \quad \text{etc.} \end{aligned} \tag{16}$$

Equations (11)–(13) and (16) provide the description of the deformed state at which the stability of the body is to be investigated. The equations corresponding to the first power of λ in the above system of boundary value problems are those of linear elasticity theory.

3. FORMULATION OF STABILITY PROBLEM

Let $\mathbf{u}^0$ be the displacements corresponding to an equilibrium state, and $\mathbf{v}$ be the displacements from this equilibrium state to a neighboring, geometrically admissible state. The potential energy in the equilibrium state is given by

$$\pi_0 = 1/2 \int_\tau (S_{ij}^0 \epsilon_{ij}^0)\, d\tau - \int_{S_T} T_i u_i^0\, dS. \tag{17}$$

Equilibrium requires that

$$\delta\pi_0 = \int_\tau S_{ij}^0 \delta\epsilon_{ij}\, d\tau - \int_{S_T} T_i \delta u_i\, dS = 0. \tag{18}$$

The linearized variation of eqn (2) is

$$\delta\epsilon_{ij} = \tfrac{1}{2}(\eta_{i,j} + \eta_{j,i}) + \tfrac{1}{2} u_{k,i}^0 \eta_{k,j} + \tfrac{1}{2} u_{k,j}^0 \eta_{k,i} \tag{19}$$

where $\eta_i = \delta u_i$ is any kinematically admissible displacement field. Equation (18), upon substitution of eqn (19), becomes

$$\int_\tau S^0_{ij}(\eta_{i,j} + u^0_{k,i}\eta_{k,j})\,\mathrm{d}\tau - \int_{S_T} T_i\eta_i\,\mathrm{d}S = 0 \tag{20}$$

for all admissible η_i. The potential energy in a neighboring state (identified by the additional stresses S_{ij}, etc.) is given by

$$\pi_0 + \pi = 1/2\int_\tau (S^0_{ij} + S_{ij})(\epsilon^0_{ij} + \epsilon_{ij})\,\mathrm{d}\tau - \int_{S_T} T_i(u_i^0 + v_i)\,\mathrm{d}S. \tag{21}$$

The change in the total potential energy, π, in going from the equilibrium configuration to the adjacent state, by eqn (17) and eqn (21), is then

$$\pi = \int_\tau \left(S^0_{ij}\epsilon_{ij} + \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl}\right)\mathrm{d}\tau - \int_{S_T} T_i v_i\,\mathrm{d}S. \tag{22}$$

In a similar fashion, and through the use of eqn (19), the change in the strain in passing to the adjacent state may be written as

$$\epsilon_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) + \tfrac{1}{2}u^0_{k,i}v_{k,j} + \tfrac{1}{2}u^0_{k,j}v_{k,i} + \tfrac{1}{2}v_{k,i}v_{k,j}. \tag{23}$$

Inserting eqn (23) into eqn (22) then yields

$$\pi = V_1 + V_2 + V_3 + V_4 \tag{24a}$$

in which

$$V_1 = \int_\tau S^0_{ij}(v_{i,j} + u^0_{m,i}v_{m,j})\,\mathrm{d}\tau - \int_{S_T} T_i v_i\,\mathrm{d}S \tag{24b}$$

$$V_2 = 1/2\int_\tau S^0_{ij}v_{m,i}v_{m,j}\,\mathrm{d}\tau + 1/2\int_\tau C_{ijkl}(v_{k,l} + u^0_{m,k}v_{m,l})(v_{i,j} + u^0_{p,i}v_{p,j})\,\mathrm{d}\tau \tag{24c}$$

$$V_3 = 1/2\int_\tau C_{ijkl}(v_{m,k}v_{m,l})(v_{i,j} + u^0_{p,i}v_{p,j})\,\mathrm{d}\tau \tag{24d}$$

$$V_4 = 1/8\int_\tau C_{ijkl}(v_{m,k}v_{m,l})(v_{p,i}v_{p,j})\,\mathrm{d}\tau \tag{24e}$$

where V_1, V_2, V_3, V_4 are the linear, quadratic, cubic, and quartic terms in the displacements, $\mathbf{v}$, respectively, of the change in the total potential energy. Since eqn (20) is valid for all admissible η_i, we choose in particular that $\eta_i = v_i$. Comparison of eqn (20) with eqn (24b) results in

$$\pi = V_2 + V_3 + V_4.$$

As expected, there are no linear terms, since the basic state is one of equilibrium. The equilibrium state is stable if $\pi > 0$ for all neighboring states, that is, if $V_2 > 0$ for all v_i (because V_2 dominates V_3 and V_4 for sufficiently small values of $\mathbf{v}$). The critical, or bifurcation, state is reached when V_2 becomes semi-definite, or equivalently, when

$$V_2|_{\min} = 0. \tag{25}$$

That is, the problem is to find the displacement (or buckling mode) $\mathbf{v}$ which minimizes V_2, subject to a suitable norm, and to let that minimum vanish.

4. SOLUTION TO THE BUCKLING EQUATION

The quadratic terms in the change in the total potential energy which are given in eqn (24c), can be rewritten as

$$2V_2 = \int_\tau C_{ijkl} v_{i,j} v_{k,l}\, d\tau + 2\int_\tau C_{ijkl} v_{i,j} u^0_{m,k} v_{m,l}\, d\tau + \int_\tau C_{ijkl} v_{m,i} u^0_{m,j} u^0_{p,k} v_{p,l}\, d\tau + \int_\tau S^0_{ij} v_{m,i} v_{m,j}\, d\tau \tag{26}$$

where by expanding the prebuckling stresses S_{ij} and displacements $\mathbf{u}$ as shown in eqn (7), eqn (26) becomes

$$2V_2 = A(\mathbf{v},\mathbf{v}) + \lambda[B^{(1)}(\mathbf{v},\mathbf{v}) + 2C^{(1)}(\mathbf{v},\mathbf{v})] + \lambda^2[B^{(2)}(\mathbf{v},\mathbf{v}) + 2C^{(2)}(\mathbf{v},\mathbf{v}) + D^{(1,1)}(\mathbf{v},\mathbf{v})] + \cdots \tag{27}$$

in which the quadratic functionals are given by

$$\begin{aligned} A(\mathbf{v},\mathbf{v}) &= \int_\tau C_{ijkl} v_{i,j} v_{k,l}\, d\tau \\ B^{(n)}(\mathbf{v},\mathbf{v}) &= \int_\tau S^{(n)}_{ij} v_{k,i} v_{k,j}\, d\tau \quad (n = 1,2,\ldots) \\ C^{(n)}(\mathbf{v},\mathbf{v}) &= \int_\tau C_{ijkl} v_{i,j} u^{(n)}_{m,k} v_{m,l}\, d\tau \quad (n = 1,2,\ldots) \\ D^{(m,n)}(\mathbf{v},\mathbf{v}) &= \int_\tau C_{ijkl} v_{s,i} u^{(m)}_{s,j} u^{(n)}_{p,k} v_{p,l}\, d\tau \quad (m,n = 1,2,\ldots). \end{aligned} \tag{28}$$

A convenient normalizing condition for the buckling mode $\mathbf{v}$ is

$$B^{(1)}(\mathbf{v},\mathbf{v}) = -1 \tag{29}$$

in which the negative sign is chosen so as to lead to positive values of λ. Then, with the introduction of the Lagrangian multiplier q through

$$Q \equiv V_2 + 1/2q[B^{(1)}(\mathbf{v},\mathbf{v}) + 1] \tag{30}$$

we find that a necessary condition for minimizing V_2, subject to the associated normalizing condition eqn (29), is given by

$$\begin{aligned} \delta Q \equiv{}& A(\mathbf{v},\boldsymbol{\eta}) + \lambda[B^{(1)}(\mathbf{v},\boldsymbol{\eta}) + C^{(1)}(\mathbf{v},\boldsymbol{\eta}) + C^{(1)}(\boldsymbol{\eta},\mathbf{v})] \\ &+ \lambda^2[B^{(2)}(\mathbf{v},\boldsymbol{\eta}) + C^{(2)}(\mathbf{v},\boldsymbol{\eta}) + C^{(2)}(\boldsymbol{\eta},\mathbf{v}) + D^{(1,1)}(\mathbf{v},\boldsymbol{\eta})] + \cdots \\ &+ qB^{(1)}(\mathbf{v},\boldsymbol{\eta}) = 0 \end{aligned} \tag{31}$$

for all admissible variations $\boldsymbol{\eta}$. In the development of eqn (31) we have made use of the symmetry of the bilinear forms A, $B^{(n)}$, and $D^{(1,1)}$; however $C^{(n)}$ is not symmetric, nor is $D^{(m,n)}$ for $m \neq n$ ($D^{(m,n)}$ is needed if the expansion is to be carried out beyond λ^2). Equation (31) represents a linear eigenvalue problem, in which both the eigenvalue q and associated eigenmode $\mathbf{v}$ are clearly functions of the load parameter λ. It is reasonable to assume that this functional dependence on λ is regular near $\lambda = 0$, and that expansions of the type

$$\begin{aligned} \mathbf{v} &= \mathbf{v}^{(0)} + \lambda\mathbf{v}^{(1)} + \lambda^2\mathbf{v}^{(2)} + \cdots \\ q &= q_0 + \lambda q_1 + \lambda^2 q_2 + \cdots \end{aligned} \tag{32}$$

are therefore admissible. In eqns (32), the terms $\mathbf{v}^{(0)}$ and q_0 correspond to $\lambda = 0$, that is, to the case in which no prebuckling deformations take place. The effect of the latter is incorporated through the terms involving λ to the first power and higher. Insertion of the expansion (eqns (32)) into eqn (31) now leads to a power series in λ, with each coefficient representing a system of equations which has to be independently equated to zero. The first three coefficients (corresponding, respectively, to λ^0, λ^1, and λ^2) give rise to the following equations:

$$A(\mathbf{v}^{(0)}, \boldsymbol{\eta}) + q_0 B^{(1)}(\mathbf{v}^{(0)}, \boldsymbol{\eta}) = 0 \tag{33a}$$

$$A(\mathbf{v}^{(1)}, \boldsymbol{\eta}) + q_0 B^{(1)}(\mathbf{v}^{(1)}, \boldsymbol{\eta}) = -(1+q_1)B^{(1)}(\mathbf{v}^{(0)}, \boldsymbol{\eta}) - C^{(1)}(\mathbf{v}^{(0)}, \boldsymbol{\eta}) - C^{(1)}(\boldsymbol{\eta}, \mathbf{v}^{(0)}) \tag{33b}$$

$$\begin{aligned} A(\mathbf{v}^{(2)}, \boldsymbol{\eta}) + q_0 B^{(1)}(\mathbf{v}^{(2)}, \boldsymbol{\eta}) = & -(1+q_1)B^{(1)}(\mathbf{v}^{(1)}, \boldsymbol{\eta}) - C^{(1)}(\mathbf{v}^{(1)}, \boldsymbol{\eta}) - C^{(1)}(\boldsymbol{\eta}, \mathbf{v}^{(1)}) - B^{(2)}(\mathbf{v}^{(0)}, \boldsymbol{\eta}) \\ & - C^{(2)}(\mathbf{v}^{(0)}, \boldsymbol{\eta}) - C^{(2)}(\boldsymbol{\eta}, \mathbf{v}^{(0)}) - D^{(1,1)}(\mathbf{v}^{(0)}, \boldsymbol{\eta}) - q_2 B^{(1)}(\mathbf{v}^{(0)}, \boldsymbol{\eta}). \end{aligned} \tag{33c}$$

The problem is made definite by expanding eqn (29) in powers of λ and by postulating

$$B^{(1)}(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}) + 1 = 0 \tag{34a}$$

$$B^{(1)}(\mathbf{v}^{(0)}, \mathbf{v}^{(1)}) = 0. \tag{34b}$$

Equations (33) hold for all admissible $\boldsymbol{\eta}$. So by letting $\boldsymbol{\eta} = \mathbf{v}^{(0)}$ in eqn (33a), we obtain

$$A(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}) + q_0 B^{(1)}(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}) = 0$$

which because of eqn (34a) may be written as

$$q_0 = A(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}). \tag{35}$$

The eigenvector $\mathbf{v}^{(0)}$ and associated eigenvalue q_0 of eqn (35) is the buckling solution for the body with initial displacements neglected. However, when q_0 is obtained in this manner, it renders eqn (33b) singular. A solution will exist only if the right-hand side of eqn (33b) is subjected to a suitable orthogonality condition. This condition is obtained by letting $\boldsymbol{\eta} = \mathbf{v}^{(1)}$ in eqn (33a), by letting $\boldsymbol{\eta} = \mathbf{v}^{(0)}$ in eqn (33b), and by subtracting the resulting equations, which yields

$$q_1 = -1 + 2C^{(1)}(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}) \tag{36}$$

which is subject to the normalization of eqn (34a). Finally, following a similar argument, by letting $\boldsymbol{\eta} = \mathbf{v}^{(2)}$ in eqn (33a), and letting $\boldsymbol{\eta} = \mathbf{v}^{(0)}$ in eqn (33c), and then subtracting the resulting equations from each other, we obtain

$$\begin{aligned} q_2 = & \; C^{(1)}(\mathbf{v}^{(0)}, \mathbf{v}^{(1)}) + C^{(1)}(\mathbf{v}^{(1)}, \mathbf{v}^{(0)}) \\ & + B^{(2)}(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}) + 2C^{(2)}(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}) + D^{(1,1)}(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}) \end{aligned} \tag{37}$$

subject as before to eqns (34).

It is noted that eqns (35) and (36) involve only the base function $\mathbf{v}^{(0)}$. In contrast to q_0 and q_1, however, the formula for q_2 (eqn (37)) involves also the function $\mathbf{v}^{(1)}$, which is obtained by solving eqn (33b). The latter now admits a solution since the secular term on the right side has been removed through eqn (36). The solution, however, is not unique. In fact, let $\bar{\mathbf{v}}^{(1)}$ be a particular solution; then, by virtue of eqn (33a), it is easy to see that

$$\mathbf{v}^{(1)} = \bar{\mathbf{v}}^{(1)} + \alpha \mathbf{v}^{(0)} \tag{38}$$

is also a solution of eqn (33b), regardless of the value of α. The specific value of α to be chosen is determined by eqn (34b), which, after substitution of eqn (38), and in view of eqn (34a), yields

$$\alpha = B^{(1)}(\mathbf{v}^{(0)}, \bar{\mathbf{v}}^{(1)}). \tag{39}$$

We now return to eqn (31) and select $\eta = \mathrm{v}$. When the resulting equation is compared with eqn (27) and when eqn (34a) is taken into consideration, we obtain

$$V_2|_{min} = 1/2q = 1/2(q_0 + \lambda q_1 + \lambda q_2^2 + \cdots) \tag{40}$$

in which the second equality follows from eqn (32). The bifurcation condition eqn (25) therefore identifies the critical load parameter λ_{cr} as the root of the critical equation

$$q_0 + \lambda q_1 + \lambda^2 q_2 + \cdots = 0. \tag{41}$$

It is interesting to note that if the quadratic term in eqn (41) is deleted and if q_1 in eqn (36) is approximated by $q_1 = -1$, then

$$\lambda_{cr}^{(0)} = q_0 \tag{42}$$

which represents the classical approximation and disregards the prebuckling deformations. A better approximation is given by

$$\lambda_{cr}^{(1)} = -q_0/q_1 \tag{43}$$

while the inclusion of the quadratic term leads to

$$\lambda_{cr}^{(2)} = \frac{1}{2q_2}[-q_1 \pm \surd(q_1^2 - 4q_0q_2)] \tag{44}$$

in which the root has to be selected so as to let $\lambda_{cr}^{(2)}$ approximate $\lambda_{cr}^{(1)}$. In these formulas, the values of q_0, q_1, and q_2 are given in eqns (35), (36) and (37), respectively.

5. DISCRETE ELEMENT FORMULATION

The development of the previous section follows closely the pattern set by Masur and Schreyer[9]. In this section the equations will be converted into a form suitable for finite element applications. To begin, we subdivided the domain τ into n_e subdomains, or finite elements, $\tau^{(e)}$, interconnected by nodes. The displacement fields are given in terms of shape functions $\phi_{iM}^{(e)}(\mathbf{A})$ and nodal displacements U_M, $M = 1$ to n, by

$$u_i = \phi_{iM}^{(e)} L_{MN}^{(e)} U_N = \Phi_{iN} U_N \quad (\text{sum on } e, e = 1 \text{ to } n_e) \tag{45}$$

where $L_{MN}^{(e)}$ is the connectivity matrix; upper case subscripts denote nodal variables and repeated upper case subscripts imply a summation from 1 to n, the degrees of freedom of the discrete system.

As is well known, the shape functions must be constructed so that if derivatives of order p appear in energy expressions, then the lowest $(p-1)$th derivatives are continuous both within and across elements. In addition, the shape functions must be complete; a sufficient condition for completeness is the capability of describing rigid body motions and a constant strain state, Oden[10].

The same shape functions will be used for both prebuckling displacements and the buckling mode. In the finite element formulation, these two displacement fields are then distinguished by the nomenclature for the nodal displacements; U_N denotes the prebuckling nodal displacements, while V_N denotes the buckling mode nodal displacements.

The first step in the analysis is the determination of the first and second order approximations for the prebuckling displacements. These equations are obtained by considering a discrete form of eqn (20). Inserting eqn (45) into eqn (20), we obtain

$$H_M \int_\tau S_{ij}(\Phi_{iM,j} + \Phi_{kN,i}\Phi_{kM,j}U_N)\,d\tau - H_M \int_{S_T} \lambda T_i \Phi_{iM}\,dS = 0 \tag{46}$$

where H_M are the virtual nodal displacements. The finite element counterpart of eqns (16) is

$$S_{ij}^{(1)} = C_{ijkl}\Phi_{kM,l}U_M{}^{(1)}$$
$$S_{ij}^{(2)} = C_{ijkl}(\Phi_{kM,l}U_M{}^{(2)} + 1/2\Phi_{pM,k}U_M{}^{(1)}\Phi_{pN,l}U_N{}^{(1)}). \tag{47}$$

Expanding the stresses and displacements as given by eqn (7), we obtain

$$H_M \int_\tau [\Phi_{iM,j} + \Phi_{kN,i}\Phi_{kM,j}(\lambda U_N{}^{(1)} + \lambda^2 U_N{}^{(2)})]$$
$$\times [\lambda C_{ijkl}\Phi_{kP,l}U_P{}^{(1)} + \lambda^2 C_{ijkl}(\Phi_{kP,l}U_P{}^{(2)} + 1/2\Phi_{pP,k}\Phi_{pR,l}U_p{}^{(1)}U_R{}^{(1)})]\, d\tau - H_M \int_{S_T} \lambda T_i\Phi_{iM}\, dS = 0. \tag{48}$$

Since each coefficient of this polynomial in λ must individually vanish and since the values of H_N are arbitrary, it follows that

$$K_{MP}^{(E)}U_P{}^{(1)} = F_M{}^{(1)} \tag{49a}$$
$$K_{MP}^{(E)}U_P{}^{(2)} = F_M{}^{(2)} \tag{49b}$$

where

$$K_{MP}^{(E)} = \int_\tau C_{ijkl}\Phi_{iM,j}\Phi_{kP,l}\, d\tau = K_{PM}^{(E)} \tag{50}$$
$$F_M{}^{(1)} = \int_{S_T} T_i\Phi_{iM}\, dS \tag{51}$$
$$F_M{}^{(2)} = -1/2\int_\tau C_{ijkl}\Phi_{iM,j}u_{p,k}^{(1)}u_{p,l}^{(1)}\, d\tau - \int_\tau C_{lkij}\Phi_{pM,l}u_{p,k}^{(1)}u_{i,j}^{(1)}\, d\tau. \tag{52}$$

The matrix $K_{MN}^{(E)}$ is the linear elastic stiffness matrix, so eqn (49a) is the standard linear equation of matrix structural analysis. Equation (49b) is identical to eqn (49a) except that the right hand side includes the "pseudoforces" corresponding to the first approximation of the nonlinear terms.

The second step in this procedure is the determination of the buckling mode expansions, $\mathbf{v}^{(0)}$ and $\mathbf{v}^{(1)}$, by eqns (33). For these purposes, we note that according to eqn (28)

$$A(\mathbf{v}, \boldsymbol{\eta}) = H_M K_{MN}^{(E)} V_N \tag{53}$$

while

$$B^{(1)}(\mathbf{v}, \boldsymbol{\eta}) = H_M K_{MN}^{(G)} V_N \tag{54}$$

where

$$K_{MN}^{(G)} = \int_\tau S_{ij}^{(1)}\Phi_{kN,i}\Phi_{kM,j}\, d\tau. \tag{55}$$

The matrix $K_{MN}^{(G)}$ is the well known initial stress matrix, Przemieniecki[11].

From the arbitrariness of H_N and the above equations, it follows that the discrete form of eqn (33a) is

$$K_{MN}^{(E)}V_N{}^{(0)} + q_0 K_{MN}^{(G)}V_N{}^{(0)} = 0. \tag{56}$$

This is the standard eigenvalue problem for the determination of the buckling load with prebuckling displacements neglected.

In a similar manner, it can be shown that the discrete form of eqn (33b) is

$$K_{MN}^{(E)}V_N{}^{(1)} + q_0 K_{MN}^{(G)}V_N{}^{(1)} =$$
$$-(1+q_1)\int_\tau S_{ij}^{(1)}v_{k,i}^{(0)}\Phi_{kM,j}\, d\tau - \int_\tau C_{ijkl}u_{m,k}^{(1)}(v_{i,j}^{(0)}\Phi_{mM,l} + v_{m,l}^{(0)}\Phi_{iM,j})\, d\tau \tag{57}$$

where q_1 has been evaluated by eqn (36). These linear algebraic equations, like their continuous counterpart, are singular. If q_0 is a single root of the characteristic equation, eqn (56), then the rank of this system of algebraic equations is $n-1$, and its solution may be expressed in terms of a single parameter by an elimination procedure. This parameter is then evaluated by the normality condition, eqn (34b), which in finite element form is

$$K^{(G)}_{MN}V_M{}^{(0)}V_N{}^{(1)}=0. \tag{58}$$

The equations developed here can be simplified for beam elements in two dimensional structures by assuming that in the buckling mode the beam elements are inextensible, i.e. that the buckling mode consists primarily of rotations and translations of the elements. These simplifications will be developed in the following.

Consider an Euler–Bernoulli beam of symmetrical cross section with its neutral surface coincident with a local x-axis. The axial displacements in a cross section are then given by

$$v_x=\bar{v}_x-yv_{y,x} \tag{59}$$

in which $\bar{v}_x$ and v_y are functions of x alone and where the bar designates the axial displacement of the midplane, similarly for $\mathbf{u}$. Both $\bar{v}_x$ and v_y are expanded in a power series in λ.

If Poisson's ratio is taken to be zero, then the pertinent terms in eqn (28) can be simplified as follows

$$\begin{aligned}
C^{(1)}(\mathbf{v}^{(0)},\mathbf{v}^{(0)})&=EA\int_L \bar{v}^{(0)}_{x,x}v^{(0)}_{y,x}u^{(1)}_{y,x}\,\mathrm{d}x\\
C^{(1)}(\mathbf{v}^{(0)},\mathbf{v}^{(1)})&=EA\int_L \bar{v}^{(0)}_{x,x}v^{(1)}_{y,x}u^{(1)}_{y,x}\,\mathrm{d}x\\
C^{(1)}(\mathbf{v}^{(1)},\mathbf{v}^{(0)})&=EA\int_L \bar{v}^{(1)}_{x,x}v^{(0)}_{y,x}u^{(1)}_{y,x}\,\mathrm{d}x\\
D^{(1,1)}(\mathbf{v}^{(0)},\mathbf{v}^{(0)})&=EA\int_L [v^{(0)}_{y,x}u^{(1)}_{y,x}]^2\,\mathrm{d}x\\
C^{(2)}(\mathbf{v}^{(0)},\mathbf{v}^{(0)})&=EA\int_L \bar{v}^{(0)}_{x,x}v^{(0)}_{y,x}u^{(2)}_{y,x}\,\mathrm{d}x.
\end{aligned} \tag{60}$$

Here the integral extends over the length of the element. Repeated use is made of the fact that for a beam, the only quadratic term which need be retained is rotational; the quadratic extensional terms can be assumed to be negligible by comparison. The additional axial strain during buckling is given through eqn (23) as

$$\epsilon_{xx}=\bar{v}_{x,x}+u_{y,x}v_{y,x} \tag{61}$$

where the second order terms in $\mathbf{v}$ have been omitted. The expansions for $\mathbf{u}$ and $\mathbf{v}$, eqns (7) and (32), respectively, yield

$$\epsilon_{xx}=\bar{v}^{(0)}_{x,x}+\lambda(\bar{v}^{(1)}_{x,x}+u^{(1)}_{y,x}v^{(0)}_{y,x})+\cdots. \tag{62}$$

Hence, if we assume that the extension of the neutral axis in the buckling mode vanishes, then each coefficient of the above polynomial in λ must also vanish, so that

$$\begin{aligned}
\bar{v}^{(0)}_{x,x}&=0\\
\bar{v}^{(1)}_{x,x}+u^{(1)}_{y,x}v^{(0)}_{y,x}&=0
\end{aligned} \tag{63}$$

these conditions, when applied to eqn (60) yield

$$C^{(1)}(\mathbf{v}^{(0)},\mathbf{v}^{(0)})=C^{(1)}(\mathbf{v}^{(0)},\mathbf{v}^{(1)})=C^{(2)}(\mathbf{v}^{(0)},\mathbf{v}^{(0)})=0$$

$$C^{(1)}(\mathbf{v}^{(1)}, \mathbf{v}^{(0)}) = -AE\int_L (u^{(1)}_{y,x} v^{(0)}_{y,x})^2\, dx. \tag{64}$$

By using the above, it can be shown that eqns (36) and (37) reduce to

$$\begin{aligned} q_1 &= -1 \\ q_2 &= B^{(2)}(\mathbf{v}^{(0)}, \mathbf{v}^{(0)}). \end{aligned} \tag{65}$$

The axial thrust is the only contribution to S_{ij} retained in $B^{(2)}$ and the initial stress stiffness, $K^{(G)}_{MN}$. The critical load is then again given by eqn (44). The effect of the prebuckling displacements in this case enters entirely through the term $S^{(2)}_{ij}$ in $B^{(2)}(\mathbf{v}^{(0)}, \mathbf{v}^{(0)})$, which are the stresses associated with the second order approximation to the prebuckling displacements.

The procedure described here has been programmed for beam elements in two dimensional structures. A standard beam element with a cubic transverse and linear axial displacement field was used; the displacement field and elastic and initial stress stiffness matrices may be found in Przemieniecki[11]. A flow chart of the procedure is given in Fig. 1.

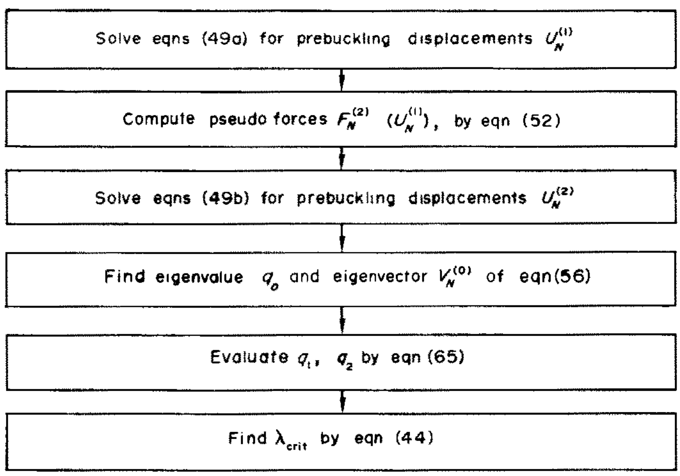

Fig. 1. Flowchart of computational procedure for beam elements in two dimensional structures.

6. RESULTS

The planar buckling of a shallow circular arch was chosen as the illustrative example. The dimensions and nomenclature for the arch are shown in Figure 2. For a half-angle of $\alpha = 10^{\circ}$, the series of problems indicated in Table 1 was solved. The entire arch was modelled by rectilinear

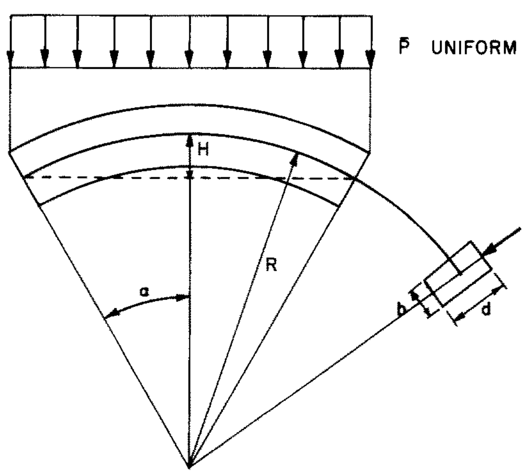

Fig. 2. Clamped circular arch under uniform load.

Table 1. Results for critical load for arch

Geometry				Results				Previous Results	
Radius	H Rise	Span	$\Lambda = \frac{2H}{\text{depth}}$	λ_0, Euler	λ_2 (second approx.)	No. Elem.	$\frac{\lambda_0-\lambda_2}{\lambda_0} \times 100$	Masur, E., Schreyer, H.[9]	Exact Schreyer, H.[12]
790"	12"	274"	6	2.26	1.87	10	17.3	1.96	1.67
1050	16	365	8	2.17	1.94 1.92 1.92 1.92 1.92 1.88	5 10 15 20 25 50	11.5	1.98	1.86
1315	20	457	10	2.12	1.95	10	8.0	1.99	1.93
1580	24	549	12	2.10	1.97	10	6.2	2.01	1.97
1840	28	639	14	2.08	2.00	10	4.1	2.02	1.99

elements of equal arc length. The load is normalized by

$$\bar{p} = \left(\frac{2H}{\pi^2}\right)\left(\frac{12R^2}{Ed^3}\right)p \tag{66}$$

according to Schreyer[12]. The geometric parameters for all test cases were chosen so the mode of instability was asymmetric. As can be seen from Table 1, including the effect of the prebuckling displacements reduces the bifurcation load as compared to the Euler solution in all cases. This effect is more substantial for the shallower arches. At large values of Λ, the proposed technique is in good agreement with the exact solution. For shallower arches, the agreement is not as good. This seems reasonable, for the magnitude of the displacements before buckling approaches the rise of the arch. The correct description of the nonlinear fundamental path becomes even more important for very shallow arches and a partial series expansion for these displacements may not be sufficient. The disparity in the results may be attributed to the fact that unlike Masur and Schreyer[9], in this investigation no assumption was made as to the constancy of the individual contributions of axial thrust given by eqn (7). Although the first order prebuckling axial force, which is the solution to eqn (49a), varies by less than 0·1 percent over the span, the total axial force, including second order terms, was found to vary along the span by 2 to 6%, depending on the shallowness of the arch.

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